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Z_3-symmetric conformal algebra from KdV-type equation

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Abstract. We construct, by using the KdV–Virasoro connection, Z_3-symmetric conformal algebra from the KdV-type equation; the Boussinesq equation. Also we find that the Miura transformation of the Boussinesq equation is equivalent to the definition of the energy-momentum tensor, T, and the spin-3 field, W.

1. Introduction

Recently much attention has been paid to the quantisation of the Liouville equation which is connected with the quantisation of the strings by using the path integral formalism [1]. Although this approach has not been completed, it gives some interesting relations between the Virasoro algebra and the integrable systems in two-dimensional spacetime, in particular to the Korteweg-de Vries equation (KdV) [2].

In general, the Virasoro algebra associated with the current algebra (or a Kac-Moody algebra) [3] is constructed by using the Sugawara mechanism [4]. Formally infinite Virasoro generators are the Laurent expansion of the square of currents. On the other hand, a quietly similar role in the KdV equation is played by the Miura transformation [5] that relates the KdV solution, u, and the modified KdV (MKdV) solution, v, in the form $u = v_x - v^2$. This transformation enables one to derive the Poisson bracket of the KdV system from that of the MKdV system that is quietly similar to the commutation relation among currents in the current algebra. An appropriate Fourier expansion of this induced Poisson bracket of the KdV system leads to the classical form of the Virasoro algebra [2].

Such a connection has been extended to a supersymmetric system, i.e. the super-KdV equation [6]. The super-Miura transformation is derived from the super-Virasoro algebra and vice versa [7]. It allows one to introduce the KdV or MKdV variables as a tool to attack the various problems of string theory [8], and quantisation of the MKdV equation [9] or the Liouville equation [1].

In this paper, we will extend the KdV–Virasoro connection to the connection of Z_3-symmetric conformal algebra [10] and integrable equations having the third-order differential spectral equation as one of the Lax pair [11]; for examples, see [12], the Sawada and Kotera equation [13], the Kupershmidt equation [14] and so on. Further, we show that the form of the energy-momentum tensor, T, and the spin-3 field, W, for two-component massless free bosonic fields is equivalent to the Miura transformation for the Boussinesq equation.

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2. The Z_3-symmetric conformal algebra

We wish here to extend the KdV system to some integrable equations having the third-order differential spectral equation as one of their Lax pair; for example, the Boussinesq equation, the Kupershmidt equation, and so on [15, 16].

To associate them with the Z_3-symmetric conformal algebra, let us start from the third-order differential spectral equation which looks like the extension of a Schrödinger equation,

$$(\partial^3 + V_1\partial + V_0)\Psi = \lambda^3\Psi. \tag{1}$$

We introduce λ-weight on Ψ, ∂, V_1 and V_0 as 0, +1, +2 and +3, respectively. Equation (1) may be rewritten in terms of new variables v_1, v_2 and v_3 of a λ-weight 1,

$$(\partial + v_3)(\partial + v_2)(\partial + v_1)\Psi = \lambda^3\Psi. \tag{2}$$

The constraint equation is given by

$$v_1 + v_2 + v_3 = 0 \tag{3}$$

and the transformation rules are

$$V_1 = -[2v_{3x} + v_{2x} + \tfrac{1}{2}(v_1^2 + v_2^2 + v_3^2)] \tag{4}$$

$$V_0 = (v_{1x} + v_1v_2)_x + v_3(v_{1x} + v_1v_2). \tag{5}$$

Miura transformations can be derived from the transformation rules (4) and (5) for appropriate solutions to the constraint equation (3) [11]. These problems will be discussed explicitly at the end of this section.

We define the Poisson bracket among v_i:

$$\{v_i(x), v_j(y)\} \equiv \begin{cases} 4\pi d_3\delta'(x-y) & \text{if } i=j \\ -2\pi d_3\delta'(x-y) & \text{if } i\neq j. \end{cases} \tag{6}$$

This choice is almost unique in order to define the Hamiltonian structure of the system (1) for the general solution to the constraint equation (3). After some calculations, we obtain the Poisson bracket relations among V_0 and V_1:

$$\{V_1(x), V_1(y)\} = -2\pi d_3(6\delta''' + 6V_1\delta' + 3V_{1x}\delta) \tag{7}$$

$$\{V_1(x), V_0(y)\} = 2\pi d_3[3\delta'''' + 3V_1\delta'' + 3(2V_{1x} - 3V_0)\delta + 3(V_{1xx} - 2V_{0x})\delta] \tag{8}$$

$$\{V_0(x), V_1(y)\} = -2\pi d_3(3\delta'''' + 3V_1\delta'' + 9V_0\delta' + 3V_{0x}\delta) \tag{9}$$

$$\begin{aligned}\{V_0(x), V_0(y)\} = 2\pi d_3[&2\delta''''' + 4V_1\delta''' + 6V_{1x}\delta'' \\ &+ (2V_1^2 + 6V_{1xx} - 6V_{0x})\delta' + ((V_1^2)_x - 3V_{0xx} + 2V_{1xxx})\delta].\end{aligned} \tag{10}$$

One can easily check that the above Poisson brackets are related to the recursion operator $(L_3^+)_{ij}$ of equation (1) defined by [15],

$$\{V_i(x), V_j(y)\} = 18\pi d_3(L_3^+)_{ij}\delta'(x-y) \tag{11}$$

where

$$\begin{aligned}(L_3^+)_{01} &= \tfrac{1}{3}\partial^3 + \tfrac{1}{3}V_1\partial + V_0 + \tfrac{1}{3}V_{0x}\partial^{-1} \\ (L_3^+)_{11} &= \tfrac{2}{3}\partial^2 + \tfrac{2}{3}V_1\partial + \tfrac{1}{3}V_{1x}\partial^{-1} \\ (L_3^+)_{10} &= -\tfrac{1}{3}\partial^3 - \tfrac{1}{3}\partial(V_1\cdot) + \tfrac{1}{3}V_0 - \tfrac{1}{3}\partial(V_{1x}\partial^{-1}\cdot) + \tfrac{2}{3}\partial(V_0\partial^{-1}\cdot) \\ (L_3^+)_{00} &= -\tfrac{2}{9}\partial^4 - \tfrac{2}{9}V_1\partial^2 - \tfrac{2}{9}\partial^2(V_1\cdot) + \tfrac{2}{3}V_{0x} - \tfrac{2}{9}V_1^2 + \tfrac{1}{3}V_{0xx}\partial^{-1} - \tfrac{2}{9}V_1V_{1x}\partial^{-1} - \tfrac{2}{9}\partial^2(V_{1x}\partial^{-1}\cdot).\end{aligned} \tag{12}$$

It is generally required that the central term $\delta'''''(x-y)$ in the Poisson brackets (8) and (9) be eliminated by introducing new variables as

$$U(x) \equiv V_1(x)$$
$$W(x) \equiv V_0(x) - \tfrac{1}{2} V_{1x}(x). \tag{13}$$

We then obtain simple Poisson brackets among them as

$$\{U(x), U(y)\} = -2\pi d_3(6\delta''' + 6U\delta' + 3U_x\delta) \tag{14}$$
$$\{U(x), W(y)\} = -2\pi d_3(9W\delta' + 6W_x\delta) \tag{15}$$
$$\{W(x), U(y)\} = -2\pi d_3(9W\delta' + 3W_x\delta) \tag{16}$$
$$\{W(x), W(y)\} = 2\pi d_3[\tfrac{1}{2}\delta''''' + \tfrac{5}{2}U\delta''' + \tfrac{15}{4}U_x\delta'' + (\tfrac{9}{4}U_{xx} + 2\Lambda)\delta' + (\tfrac{2}{4}U_{xxx} + \Lambda_x)\delta] \tag{17}$$

where

$$\Lambda(x) = [U(x)]^2. \tag{18}$$

We introduce the following Fourier transformations and commutator $[\ ,\ ] = \mathrm{i}\{\ ,\ \}$:

$$U(x) = -\frac{24}{c}\sum_m (L_m - \tfrac{1}{3}\delta_{m0})\,\mathrm{e}^{-imx}$$
$$W(x) = a\sum_m W_m\,\mathrm{e}^{-imx} \tag{19}$$

with $d_3 = 8/c$ and $a^2 = 12\cdot 5!/c^2$. Finally, we obtain the Z_3-symmetric conformal algebra [10],

$$[L_m, L_n] = (m-n)L_{m+n} + \tfrac{1}{12}c(m^3 - m)\delta_{m+n,0} \tag{20}$$
$$[L_m, W_n] = (2m-n)W_{m+n} \tag{21}$$
$$\begin{aligned}[W_m, W_n] = {} & (c/3\cdot 5!)(m-4)(m^2-1)m\delta_{m+n,0} + b^2(m-n)\Lambda_{m+n} \\ & + (m-n)[\tfrac{1}{15}(m+n+2)(m+n+3) - \tfrac{1}{6}(m+2)(n+2)]L_{m+n}\end{aligned} \tag{22}$$

where

$$b^2 = 16/5c \qquad \text{and} \qquad \Lambda_m = \sum_k L_k L_{m-k}.$$

In [10], b^2 and Λ_m are defined in a slightly different way as

$$b^2 = 16/(22+5c) \qquad \text{and} \qquad \Lambda_m = \sum_k :L_k L_{m-k}: + \tfrac{1}{5}X_m L_m$$

where

$$X_{2m} = (1+m)(1-m) \qquad \text{and} \qquad X_{2m+1} = (m+2)(1-m).$$

It is conjectured that the differences between our case and [10] in b^2 and Λ_m arise from the fact that our case rests on the classical framework. From now on, we concentrate our attention on the connection between the Z_3-symmetric conformal algebra and integrable systems. The Z_3-symmetric conformal algebra has a Virasoro algebra as its subalgebra. In order to restrict the full algebra to its subalgebra, we need solutions to the constraint equation (3). In general two kinds of solution are accepted as follows.

(i) $v_1 = 0$, $v_2 = -v_3 = v$.

Miura transformation (equation (4)):

$$U = v_x - v^2. \tag{23}$$

Spectral equation (equation (1)):

$$(\partial^3 + U\partial)\Psi = \lambda^3\Psi. \tag{24}$$

Non-linear evolution equation for U [13]:

$$U_t = U_{xxxxx} + 5UU_{xxx} + 5U_xU_{xx} + 5U^2U_x. \tag{25}$$

(ii) $v_2 = 0$, $v_3 = -v_1 = v$.
Miura transformation:

$$P \equiv V_0 = -v_x + \tfrac{1}{2}v^2. \tag{26}$$

Spectral equation:

$$(\partial^3 + 2P\partial + P_x)\Psi = \lambda^3\Psi. \tag{27}$$

Non-linear evolution equation for P [14]:

$$P_t = P_{xxxxx} + 10PP_{xxx} + 25P_xP_{xx} + 20P^2P_x. \tag{28}$$

These two systems satisfy the same Virasoro algebra under the Z_3-symmetric conformal algebra. Because of this property, equations (25) and (28) have the same modified equation, after the Miura transformation [11], as

$$v_t = v_{xxxxx} - 5(v_xv_{xxx} + v_{xx}^2 + v_x^3 + 4vv_xv_{xx} - v^2v_x - v^4v_x). \tag{29}$$

Let us consider the full Z_3 algebra. In this case the solution to the constraint equation (3) is given by

$$v_1 = -\sqrt{3}q_1 - q_2 \qquad v_2 = 2q_2 \qquad v_3 = \sqrt{3}q_1 - q_2.$$

The Poisson bracket (6) is changed into

$$\{q_i(x), q_j(y)\} = \pi d_3\delta_{ij}\delta'(x-y) \qquad i, j = 1, 2. \tag{30}$$

Considering (4), (5) and (13), we derive the Miura transformation as

$$U = -(2\sqrt{3}q_{1x} + 3q_2^2 + 3q_1^2) \tag{31}$$

$$W = -[q_{2xx} + 3\sqrt{3}q_1q_{2x} + \sqrt{3}q_2q_{1x} + 2q_2(3q_1^2 - q_2^2)]. \tag{32}$$

The corresponding spectral problem leads to

$$(\partial^3 + U\partial + \tfrac{1}{2}U_x + W)\Psi = \lambda^3\Psi. \tag{33}$$

We note that the above equation is a part of the Lax pair of the Boussinesq equation

$$\begin{aligned} U_t &= 2W_x \\ W_t &= -\tfrac{1}{6}(U_{xxx} + 4UU_x). \end{aligned} \tag{34}$$

After the Miura transformation, we obtain the modified Boussinesq equation

$$\begin{aligned} \sqrt{3}q_{1t} &= q_{2xx} + 2\sqrt{3}(q_1q_2)_x \\ q_{2t} &= -\frac{1}{\sqrt{3}}q_{2xx} - (q_2^2 - q_1^2)_x. \end{aligned} \tag{35}$$

It should be emphasised that, under the change of variables $q_i = \alpha\partial_z\phi_i$, the Miura transformation of the Boussinesq equation is equivalent to the definition of the energy-momentum tensor, T, and spin-3 field, W, for a two-component massless free scalar field in [10].

3. Conclusions

We construct, by the KdV-Virasoro connection, the classical Z_3-symmetric conformal algebra from the Boussinesq equation. Also we find that the Miura transformation of the Boussinesq equation is equivalent to the definitions of the energy-momentum tensor, T, and spin-3 field, W, in [10].

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